ASYMPTOTIC STUDY OF THE TWO-DIMENSIONAL PROBLEM OF ELASTIC IMPACT OF BARS

PMM Vol. 32, No. 3, 1968, pp. 467-471

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(Received June 3, 1966)

A solution of the two-dimensional problem of elastic impact of semi-infinite bars is presented herein. The solution is valid for all time t from the instant of impact. The behavior of this solution is investigated for $t \to \infty$. The results of the investigation substantiate the applicability of the one-dimensional approximation to the impact problem.

Let two plane semi-infinite bars move toward each other along the y-axis with the same speed v. At the instant of collision, one of the bars occupies the space $y > 0$, $|x| < h$, and the other the space $y < 0$, $|x| < h$, where $2h$ is the thickness of the bars.

The speeds of propagation of longitudinal waves (a) and of transverse waves (b) in one bar are taken to be equal to the corresponding speeds in the other bar.

The solution of the problem is sought in the form of a dilatation function Δ and the rotation $\omega/2$; i.e., Δ is the divergence of the displacement vector and ω is the curl of this vector. The functions Δ and ω must satisfy the wave Eqs.

$$\frac{\partial^2 \Delta}{\partial t^2} = a^2 \left(\frac{\partial^2 \Delta}{\partial x^2} + \frac{\partial^2 \Delta}{\partial y^2} \right), \qquad \frac{\partial^2 \omega}{\partial t^2} = b^2 \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right)$$

the initial conditions for $t = 0$

$$\Delta = \omega = \frac{\partial \omega}{\partial t} = 0, \qquad \frac{\partial \Delta}{\partial t} = -2 \frac{v}{a} \delta(y)$$

and the boundary conditions on the lateral surfaces of the bar $x = h$.

$$\frac{\partial^2 \Delta}{\partial x^2} + (1 - 2\beta^2) \frac{\partial^2 \Delta}{\partial y^2} - 2\beta^4 \frac{\partial^2 \omega}{\partial x \, \partial y} = 0, \quad 2 \frac{\partial^2 \Delta}{\partial x \, \partial y} + \beta^2 \left(\frac{\partial^2 \omega}{\partial x^2} - \frac{\partial^2 \omega}{\partial y^2} \right) = 0$$

where $\beta = b/a$.

The last relations are obtained by taking the second derivative with respect to time of the boundary conditions expressed in terms of the components of the displacement vector and then eliminating the time derivatives obtained with the help of the equations of motion of the elastic body.

As follows from the initial conditions, the line $y = 0$ is a line of discontinuity. For $t > 0$ the breaking up of this discontinuity causes a longitudinal wave ($|y| < at$) and two unloading waves from the lateral surfaces of the bar ($|x| < h - \sqrt{a^2 t^2 - y^2}$). For $t > 2h/a$, the waves of unloading reach the opposite lateral surfaces and give rise to reflected waves, the number of which increases rapidly with time.

The general solution is represented in the form of the sum of all the waves

$$\Delta(x, y, t) = \Delta_0(x, y, t) + \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} [\Delta_{nk}^{+}(x, y, t) + \Delta_{nk}^{-}(x, y, t)]$$

$$\omega(x, y, t) = \omega_0(x, y, t) + \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} [\omega_{nk}^{+}(x, y, t) + \omega_{nk}^{-}(x, y, t)]$$

where the following notation has been introduced. The longitudinal wave has the subscript 0. The unloading waves bear the double subscript 00. Reflected waves are also denoted by double subscripts; a reflected wave arising from the incidence of a longitudinal wave on a lateral surface has the first subscript increased by unity over that of the incident wave while the second script is the same as for the incident wave. A reflected wave arising from an incident transverse wave is marked by increasing the second script by unity and leaving the first script unchanged. In addition, waves moving along the positive direction of the x-axis are marked with a superior plus sign, and those moving in the reverse direction by a superior minus sign. This notation is sufficient for the classification of all the reflected waves.

It is assumed that the functions $\Delta_{nk}^{\pm}(x, y, t)$ and $\omega_{nk}^{\pm}(x, y, t)$ are equal to zero if the corresponding wave has not yet come into existence at time t. By virtue of this, the upper limits of the summations in the general solution may be taken as infinity, even though the number of reflected waves is actually finite for all finite t.

It can be shown [1] that for the longitudinal wave $\Delta_0 = -v/a$, $\omega_0 = 0$. The remaining waves are sought in the forms of the real parts of functional-invariant solutions [2].

$$\Delta_{nk}^{\pm}(x, y, t) = \operatorname{Re} \Delta_{nk}^{\pm}(\theta_{nk}^{\pm}), \qquad \omega_{nk}^{\pm}(x, y, t) = \operatorname{Re} \omega_{nk}^{\pm}(\theta_{nk1}^{\pm})$$

The phases of these waves $\theta_{nk}^{\pm}$ and $\theta_{nk1}^{\pm}$ will be complex and are determined from Eqs.

$$\begin{aligned} at &= y\theta_{nk}^{\pm} + [\pm x + (2n+1)h] \sqrt{1 - (\theta_{nk}^{\pm})^2} + 2kh \sqrt{\beta^{-2} - (\theta_{nk}^{\pm})^2} \\ at &= y\theta_{nk1}^{\pm} + 2nh \sqrt{1 - (\theta_{nk1}^{\pm})^2} + [\pm x + (2k+1)h] \sqrt{\beta^{-2} - (\theta_{nk1}^{\pm})^2} \end{aligned} \qquad (1)$$

The equations which have been obtained follow from the self-similarity of the unloading waves [1] and from the condition of equality of the phases of the incident and reflected waves on a surface of reflection.

Substitution of the functional-invariant solutions into the boundary conditions leads to recurrence relations for the functions $d\Delta_{nk}^{\pm}/d$ and $d\omega_{nk}^{\pm}/d$ (to simplify the notation, the scripts on θ will generally be omitted). With the aid of these relations the final Eqs. are derived; these have the form

$$\frac{d\Delta_{nk}^{\pm}}{d\theta} = [F_{nk}(A) - F_{n,k-1}(A)] \frac{d\Delta_{00}^{\pm}}{d\theta}, \qquad \frac{d\omega_{nk}^{\pm}}{d\theta} = [F_{nk}(A) - F_{n-1,k}(A)] \frac{d\omega_{00}^{\pm}}{d\theta}$$

$$F_{nk}(A) = \sum_{s=0}^{\infty} (-1)^{k-s} \binom{k}{s} \binom{n+k-s}{k} A^{n+k-2s}$$

$$A = \frac{-(a^2 - 2b^2\theta^2)^2 + 4b^4\theta^2 \sqrt{1-\theta^2} \sqrt{\beta^{-2} - \theta^2}}{(a^2 - 2b^2\theta^2)^2 + 4b^4\theta^2 \sqrt{1-\theta^2} \sqrt{\beta^{-2} - \theta^2}}$$

The functions $\Delta_{00}^{\pm}$ and $\omega_{00}^{\pm}$ were obtained in [1] and, in the notation adopted here, are

$$\frac{d\Delta_{00}^{\pm}}{d\theta} = \frac{-2iv(a^2 - 2b^2)(a^2 - 2b^2\theta^2)}{\pi a [(a^2 - 2b^2\theta^2)^2 + 4b^4\theta^2 \sqrt{1-\theta^2} \sqrt{\beta^{-2} - \theta^2}](1 - \theta^2)}$$

$$\frac{d\omega_{00}^{\pm}}{d\theta} = \frac{\pm 4iva\theta(a^2 - b^2)}{\pi [(a^2 - 2b^2\theta^2)^2 + 4b^4\theta^2 \sqrt{1-\theta^2} \sqrt{\beta^{-2} - \theta^2}] \sqrt{1-\theta^2}}$$

Thus, the solution for the reflected waves reduces to the quadratures

$$\Delta_{nk}^{\pm}(\theta_{nk}^{\pm}) = \int_{\theta_*}^{\theta_{nk}^{\pm}} [F_{nk}(A) - F_{n,k-1}(A)] \Delta_{00}' \, d\theta \qquad \left(\Delta_{00}' = \frac{d\Delta_{00}^{\pm}}{d\theta}\right)$$

$$\omega_{nk}^{\pm}(\theta_{nk1}^{\pm}) = \pm \int_{\theta^*}^{\theta_{nk1}^{\pm}} [F_{nk}(A) - F_{n-1,k}(A)]\, \omega_{00}'\, d\theta \qquad \left(\omega_{00}' = \pm \frac{d\omega_{00}^{\pm}}{d\theta}\right)$$

The integration is carried out along an arbitrary curve lying in the half-plane $\mathrm{Im}\,\theta > 0$ and connecting the points $\theta_{nk}^{\pm}$ and θ^*, or $\theta_{nk1}^{\pm}$ and θ^*, where θ^* is an arbitrary point belonging to the wave front, i.e., $|\mathrm{Re}\,\theta^*| < 1$, $\mathrm{Im}\theta^* = 0$. The arbitrariness of this point follows from the fact that on the wave front

$$\mathrm{Re}\, F_{nk}(A)\, \Delta'_{00} = \mathrm{Re} F_{nk}(A)\, \omega'_{00} = 0$$

and, therefore, the functions $\Delta_{nk}^{\pm}(x, y, t) = \mathrm{Re}\, \Delta_{nk}^{\pm}(\theta)$ and $\omega_{nk}^{\pm}(x, y, t) = \mathrm{Re}\, \omega_{nk}^{\pm}(\theta)$ do not depend on θ^*.

The summation of the reflected waves leads to the final expressions:

$$\Delta(x, y, t) = -v/a + \Delta^+(x, y, t) + \Delta^-(x, y, t), \qquad \omega(x, y, t) = \omega^+(x, y, t) + \omega^-(x, y, t)$$

$$\Delta^{\pm}(x, y, t) = \mathrm{Re} \sum_{s=0}^{\infty} \sum_{n=s}^{\infty} \sum_{k=s}^{\infty} \int_{\theta_{n,k+1}^{\pm}}^{\theta_{nk}^{\pm}} (-1)^{k-s} \binom{n+k-2s}{k-s} \binom{n+k-s}{k-s} A^{n+k-2s} \Delta_{00}'\, d\theta \quad (2)$$

$$\omega^{\pm}(x, y, t) = \pm\, \mathrm{Re} \sum_{s=0}^{\infty} \sum_{n=s}^{\infty} \sum_{k=s}^{\infty} \int_{\theta_{n+1,k,1}^{\pm}}^{\theta_{nk1}^{\pm}} (-1)^{k-s} \binom{n+k-2s}{k-s} \binom{n+k-s}{k-s} A^{n+k-2s} \omega_{00}'\, d\theta$$

It is a peculiarity of the solution which has been obtained that the intensity of each reflected wave as $t \to \infty$ ($\theta \to \infty$) tends toward infinity, and the larger the value of $n + k$ for the wave in question, the more rapidly it does so. Therefore, to investigate the asymptotic character of the solution, it is necessary to consider that the expansion of the solution in a power series in t in the vicinity of the point $t = \infty$ can also contain divergent terms.

Let us first investigate the behavior of the equation for the phases, Eq. (1), for large t. As these equations imply, for $t \to \infty$, either $\theta \to \infty$ (the case studied in [3]), or $y \to \infty$ ($\lim y/t \neq 0$), or else $n \to \infty$ ($\lim n/t \neq 0$) and $k \to \infty$ ($\lim k/t \neq 0$). Further, from an examination of the lower limits of the summations in Eqs. (2) it follows that $\lim n/t = \lim k/t = \lim s/t$ for $t \to \infty$. Therefore, in place of n and k the new indices $p = n + k - 2s$ and $q = k - s$ are introduced. These have the following property: $\lim p/t = \lim q/t = 0$ for $t \to \infty$.

Grouping the terms which are finite for $t \to \infty$ on the right-hand side and the diverging terms on the left-hand side, and factoring out at on the left, we can express the equation for the phases in the form

$$at = \Omega_{pq}^{\pm}(\theta), \quad at = \Omega_{pq1}^{\pm}(\theta)$$

where

$$\Omega_{pq}^{\pm}(\theta) = \frac{[\pm x + h(2p - 2q + 1)]\sqrt{1 - \theta^2} + 2hq\sqrt{\beta^{-2} - \theta^2}}{\delta(\theta)}$$

$$\Omega_{pq1}^{\pm}(\theta) = \frac{2h(p - q)\sqrt{1 - \theta^2} + (\pm x + 2hq)\sqrt{\beta^{-2} - \theta^2}}{\delta(\theta)}$$

$$\delta(\theta) = 1 - Y\theta - S(\sqrt{\beta^{-2} - \theta^2} + \sqrt{1 - \theta^2}), \qquad Y = \frac{y}{at}, \qquad S = \frac{2hs}{at}$$

For $t \to \infty$, the phase $\theta \to \theta_0(S, Y)$, where $\theta_0(S, Y)$ is the root of Eq. $\delta(\theta) = 0$. The function $\theta_0(S, Y)$ does not depend on p or q.

The general solution (2) will now be investigated further. Following [3], we separate out the part of the function $\Delta^{\pm}(x, y, t)$ which depends on q:

$$M_{ps}^{\pm} = \mathrm{Re} \sum_{q=0}^{p} \int_{\theta_2^{\pm}}^{\theta_1^{\pm}} (-1)^q \binom{p}{q} A^p \Delta_{00}'\, d\theta \qquad \begin{pmatrix} \theta_1^{\pm} = \theta_{p-q+s,\,q+s}^{\pm} \\ \theta_2^{\pm} = \theta_{p-q+s,\,q+s+1}^{\pm} \end{pmatrix}$$

and expand in a power series in t:

$$M^{\pm}_{ps} = \sum_{m=-\infty}^{\infty} C^{\pm(p,s)}_{-m} t^{-m}$$

where

$$C^{\pm(p,s)}_{-m} = \mathrm{Re}\,\frac{1}{2\pi i a^m} \sum_{q=0}^{p} \int_C [(\Omega^{\pm}_{pq})^{m-1} (\Omega^{\pm}_{pq})' - (\Omega^{\pm}_{p+1,q+1})^{m-1} (\Omega^{\pm}_{p+1,q+1})'] \times$$
$$\times \left[\int_{\theta^*}^{\theta} (-1)^q \binom{p}{q} A^p \Delta'_{00}\, d\theta\right] d\theta$$

The only singular point $\theta_0 (S, Y)$ must lie inside the closed path C. This contour is traversed in the counterclockwise direction.

Integration by parts on the contour C and representation of the functions $\Omega^{\pm}_{pq}$ and $\Omega^{\pm}_{p+1,q+1}$ in the form of binomials linear in q permits us to carry out the summation with respect to q (see, for instance, [3]). We obtain as the result that

$$M^{\pm}_{ps} = \mathrm{Re}\left(\frac{2h}{at}\right)^{p+1} \frac{(-1)^p p!}{2\pi i} \int_C (\sqrt{\beta^{-2}-\theta^2} - \sqrt{1-\theta^2}) \sqrt{\beta^{-2}-\theta^2} \frac{A^p \Delta'_{00}\, d\theta}{\delta^{p+1}(\theta)} + O(t^{-p-2})$$

The summation with respect to p is then calculated

$$N_s^{\pm} = \sum_{p=0}^{\infty} \binom{p+s}{s} M^{\pm}_{ps}$$

Taking account of the expansion

$$\binom{p+s}{s} = \frac{1}{p!} S^p \left(\frac{at}{2h}\right)^p + O(t^{p-1})$$

and deforming the contour C so that it contains the root of Eq.

$$\delta(\theta) + AS(\sqrt{\beta^{-2}-\theta^2} - \sqrt{1-\theta^2}) = 0 \tag{3}$$

we are able to sum with respect to p:

$$N^{\pm}_s = \mathrm{Re}\,\frac{2h}{at}\frac{1}{2\pi i}\int_C \frac{\sqrt{\beta^{-2}-\theta^2}\,\Delta'_{00} d\theta}{\delta(\theta) + AS(\sqrt{\beta^{-2}-\theta^2} - \sqrt{1-\theta^2})} + O(t^{-2})$$

Evaluating the integral with the aid of residues, we obtain the following expression for the function $\Delta^{\pm}(x, y, t)$:

$$\Delta^{\pm}(x, y, t) = \sum_{s=0}^{\infty} N^{\pm}_s = \mathrm{Re}\,\frac{2h}{at} \sum_{s=0}^{\infty} \frac{\sqrt{\beta^{-2}-\theta^2}\Delta'_{00}}{[\delta(\theta) + AS(\sqrt{\beta^{-2}-\theta^2} - \sqrt{1-\theta^2})]'} + O(t^{-2})$$

In these relations the prime denotes a derivative with respect to θ for constant S. The quantities S and θ are related by Eq. (3).

Using the Euler-Maclaurin formula, we replace the summation with respect to s by an integration with respect to S:

$$\Delta^{\pm}(x, y, t) = \mathrm{Re}\int_0^{S^*} \frac{\sqrt{\beta^{-2}-\theta^2}\Delta'_{00}\, dS}{[\delta(\theta) + AS(\sqrt{\beta^{-2}-\theta^2} - \sqrt{1-\theta^2})]'} + O(t^{-1}) \tag{4}$$

The upper limit of the integration S^* is determined as the value of S for which $\theta_0 (S, Y) = \theta^*$. The basis for the arbitrariness of the upper limit was explained earlier; however, in Eq. (4) this arbitrariness is not evident.

A transformation of the variable of integration from S to θ is then carried out:

$$\Delta^{\pm}(x, y, t) = \mathrm{Re} \int_{Y^{-1}}^{\theta^*} \frac{\sqrt{\beta^{-2}-\theta^2}\,\Delta'_{00}\,d\theta}{\sqrt{\beta^{-2}-\theta^2}+\sqrt{1-\theta^2}-A\left(\sqrt{\beta^{-2}-\theta^2}-\sqrt{1-\theta^2}\right)} + O(t^{-1})$$

The expressions for Δ'_{00} and A are substituted, giving:

$$\Delta^{\pm}(x, y, t) = \mathrm{Re} \int_{Y^{-1}}^{\theta^*} \frac{iv\,(a^2-2b^2)\,(a^2-2b^2\theta^2)\,d\theta}{\pi a\,(1-\theta^2)\,[a^4-4b^2(a^2-b^2)\,\theta^2]} + O(t^{-1})$$

The integral which has been obtained can be represented in the form of an integral along a closed contour by completing the curve of integration by a segment of the real axis from θ^* to Y^{-1}. This is permissible inasmuch as the integrand is purely imaginary on the real axis of θ. The calculation of the contour integral by means of residues leads to the final expressions

$$\Delta(x, y, t) = -\frac{b}{\sqrt{a^2-b^2}}\frac{v}{a} + O(t^{-1}) \qquad \text{for} \quad y < 2b\sqrt{1-\beta^2}\,t$$

$$\Delta(x, y, t) = O(t^{-1}) \qquad \text{for} \quad y > 2b\sqrt{1-\beta^2}\,t$$

In an analogous way, we conclude that $\omega(x, y, t) = 0(t^{-1})$.

Thus, the interference of the reflected waves for $t \to \infty$ leads to the formation of a longitudinal wave front moving with the wave speed of the one-dimensional approximation (for the plane problem, this speed is equal to $2b\sqrt{1-\beta^2}$). Ahead of the front, the strains disappear completely; behind the front, the strains which occur correspond to the one-dimensional approximation.

The investigation which has been carried out is not applicable to points moving with the velocity of longitudinal waves a or with the velocity of Rayleigh waves, but is valid for velocities arbitrarily close to those. This circumstance and also the comparison of the energy of the wave in the one-dimensional approximation with the energy of the impact lead to the following conclusion. The longitudinal wave propagated with the speed a and the surface wave propagated at the Rayleigh speed decay with time; i.e., the relative amount of energy concentrated in these waves tends toward zero.

In conclusion, we remark that the calculation of the succeeding terms of the expansion of the general solution in power series in t leads to considerable difficulties. Therefore, in order to ascertain the rapidity of convergence, numerical calculations were carried out. These showed that on the line of contact the exact solution oscillates about the one-dimensional approximation and approaches it comparatively rapidly (after time $t \approx 10 \div 20\,h/a$).

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Translated by A.R.R.